



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

***On the Excess of the Number of Combinations in a Set
which have an Even Number of Inversions
over those which have an Odd Number.***

BY W. H. METZLER, PH. D.

1. If we are given any combination of n numbers m at a time, the combination of the remaining $n - m$ numbers is said to be the complementary with respect to n of the given combination.

Let it be understood (unless otherwise expressed) that the numbers in any combination are arranged in their natural order (order of magnitude). Let $(n|_1 m), (n|_2 m) \dots (n|_\mu m)$ denote the $\frac{n(n-1) \dots (n-m+1)}{m!} = n_m = \mu$ combinations of the numbers 1, 2, 3 ... n taken m at a time, and let $(\bar{n}|_1 m) \dots (\bar{n}|_\mu m)$ denote their complementaries. Let $(n|_a m|_1 l), (n|_a m|_2 l) \dots (n|_a m|_\lambda l)$ denote the $n_l = \lambda$ combinations of the numbers in the combination $(n|_a m)$ taken l at a time, and let $(n|_a \bar{m}|_\beta l)$ denote the combination which is the complementary with respect to m of the combination $(n|_a m|_\beta l)$, i. e., the combination of the $m - l$ numbers remaining after the numbers in the combination $(n|_a m|_\beta l)$ are taken out of the combination $(n|_a m)$. For present purposes let $(n|_a \bar{m}|_\beta l)(n|_a m|_\beta l)$ denote the combination made up of the numbers in $(n|_a \bar{m}|_\beta l)$ followed by the numbers in $(n|_a m|_\beta l)$. In contrast with this I have used elsewhere* $(n|_a \bar{m}|_\beta l)(n|_a m|_\beta l)$ to denote the com-

* Am. Jour. Math., vol. XX, No. 3.

combination of the numbers in the two combinations $(n|\overline{m}|l)_{\alpha\beta}$ and $(n|m|l)_{\alpha\beta}$ arranged in their natural order.

Let any combination having an odd number of inversions from the natural order be affected with the negative sign.

2. If k denote the number of inversions in $(n|\overline{m}|l)_{\alpha\beta}(n|m|l)_{\alpha\beta}$, then it may be easily proven that the number of inversions in $(n|\overline{m}|l)_{\alpha\beta}(n|m|l)_{\alpha\beta}$ is $l(m-l) - k$.

Therefore

$$\begin{aligned} (n|m|l)_{\alpha\beta}(n|\overline{m}|l)_{\alpha\beta} &= (-1)^{l(m-l)-2k} (n|\overline{m}|l)_{\alpha\beta}(n|m|l)_{\alpha\beta} \\ &= (-1)^{l(m-l)} (n|\overline{m}|l)_{\alpha\beta}(n|m|l)_{\alpha\beta}. \end{aligned}$$

3. The combination $(n|\overline{m}|l)_{\alpha\beta}(n|m|l)_{\alpha\beta} = \pm (n|m)_{\alpha}$ according as the number of inversions is even or odd.

Let

$$\begin{aligned} (n|\overline{m}|l)_{\alpha 1}(n|m|l)_{\alpha 1} + (n|\overline{m}|l)_{\alpha 2}(n|m|l)_{\alpha 2} + \dots + (n|\overline{m}|l)_{\alpha \lambda}(n|m|l)_{\alpha \lambda} \\ = \phi(m, l) \cdot (n|m), \end{aligned} \quad (1)$$

then will

$$(n|m|l)_{\alpha 1}(n|\overline{m}|l)_{\alpha 1} + (n|m|l)_{\alpha 2}(n|\overline{m}|l)_{\alpha 2} + \dots + (n|m|l)_{\alpha \lambda}(n|\overline{m}|l)_{\alpha \lambda} = \phi(m, m-l) \cdot (n|m)_{\alpha}.$$

$$\begin{aligned} \text{But} \quad (n|\overline{m}|l)_{\alpha\beta}(n|m|l)_{\alpha\beta} &= (-1)^{l(m-l)} (n|m|l)_{\alpha\beta}(n|\overline{m}|l)_{\alpha\beta}, \quad (\text{art. 2}) \\ \therefore \phi(m, l) &= (-1)^{l(m-l)} \phi(m, m-l). \end{aligned}$$

If $l=1$, the signs of the left-hand member of equation (1) are evidently alternately positive and negative, therefore,

$$\phi(m, 1) = 1 \text{ or } 0$$

according as m is odd or even.

It is also apparent that

$$\phi(m, m) = 1.$$

4. The set of combinations

$$(n|\overline{m}|l)_{\alpha 1}(n|m|l)_{\alpha 1}, \quad (n|\overline{m}|l)_{\alpha 2}(n|m|l)_{\alpha 2}, \quad \dots, \quad (n|\overline{m}|l)_{\alpha \lambda}(n|m|l)_{\alpha \lambda}$$

may be divided up into groups as follows :

The first group containing the first $(m-1)_{l-1}$ combinations,
 " second " " " next $(m-2)_{l-1}$ "

 " $(m-l+1)^{\text{st}}$ " " " last $(l-1)_{l-1} = 1$ "

The first number in the second part of each combination of the r^{th} group is the same, and is the r^{th} of the selection of m numbers, i. e., the r^{th} of the numbers in $(n|m)$. The first $r-1$ numbers in the first part of each combination of the r^{th} group are the same and are the first $r-1$ of the numbers in $(n|m)$. It follows from this that the signs of the combinations of the r^{th} group are the same as or the opposite to (according as $m-l-r+1$ is even or odd, there being $m-l-r+1$ numbers in the first part greater than the first number in the second part) the signs of the corresponding members of the set obtained by striking out the $r-1$ numbers common to the first part and the one number common to the second part of each combination of the group.

We have, therefore

$$\begin{aligned} \phi(m, l) &= \phi(l-1, l-1) - \phi(l, l-1) \phi(l+1, l-1) \dots \\ &+ (-1)^{m-l-r+1} \phi(m-r, l-1) + \dots + (-1)^{m-l} \phi(m-1, l-1), \end{aligned} \quad (2)$$

a reduction formula for $\phi(m, l)$.

As an immediate consequence of equation (2), we have

$$\begin{aligned} \phi(m, l) &= \phi(m-r, l) + (-1)^{m-r-l+1} \phi(m-r, l-1) \\ &\quad + (-1)^{m-l} \phi(m-1, l-1) \\ &= \phi(m-1, l) + (-1)^{m-l} \phi(m-1, l-1). \end{aligned} \quad (3)$$

5. By successive applications of equation (2), we have

$$\begin{aligned} \phi(2m, 2l+1) &= \phi(2l, 2l) - \phi(2l+1, 2l) + \dots - \phi(2m-1, 2l) \\ &= \phi(2l-1, 2l-1) \\ &\quad - \phi(2l-1, 2l-1) + \phi(2l, 2l-1) \\ &\quad + \phi(2l-1, 2l-1) - \phi(2l, 2l-1) + \phi(2l+1, 2l-1) \\ &\quad \dots \\ &\quad - \phi(2l-1, 2l-1) + \dots + \phi(2m-2, 2l-1) \\ &= \phi(2l, 2l-1) + \phi(2l+2, 2l-1) + \dots \\ &\quad + \phi(2m-2, 2l-1). \end{aligned} \quad (4)$$

Put $l = 1$; then

$$\begin{aligned}\phi(2m, 3) &= \phi(2, 1) + \phi(4, 1) + \dots + \phi(2m-2, 1) \\ &= 0. \quad (\text{art. 3}).\end{aligned}$$

Put $l = 2$, then

$$\begin{aligned}\phi(2m, 5) &= \phi(4, 3) + \phi(6, 3) + \dots + \phi(2m-2, 3) \\ &= 0.\end{aligned}$$

In this way it may be shown that

$$\phi(2m, 2l+1) = 0, \quad (l = 1, 2, \dots, \overline{m-1}). \quad (5)$$

6. From equations (3) and (5), we have

$$\begin{aligned}\phi(2m+1, 2l) &= \phi(2m, 2l) - \phi(m, 2l-1) \\ &= \phi(2m, 2l), \\ \phi(2m+1, 2l+1) &= \phi(2m, 2l+1) + \phi(2m, 2l) \\ &= \phi(2m, 2l). \\ \phi(2m+1, 2l+1) &= \phi(2m+1, 2l) = \phi(2m, 2l).\end{aligned} \quad (6)$$

It follows from this and art. 3 that

$$\phi(m, l) = \phi(m, m-l).$$

If $l = m$, then

$$\phi(m, m) = \phi(m, 0) = 1.$$

7. From equations (2), (5) and (6), we have

$$\begin{aligned}\phi(2m, 2l) &= \phi(2m-1, 2l-1) + \phi(2m-3, 2l-1) + \dots \\ &\quad + \phi(2l-1, 2l-1) \\ &= \phi(2m-1, 2l-2) + \phi(2m-4, 2l-2) + \dots \\ &\quad + \phi(2l-2, 2l-2).\end{aligned} \quad (7)$$

These properties at once suggest that

$$\phi(2m, 2l) = m_l,$$

and it may be easily proved that this is true.

If, in equation (7), we put—

1st. $l = 1$, then

$$\begin{aligned}\phi(2m, 2) &= \phi(2m-2, 0) + \phi(2m-4, 0) + \dots + \phi(0, 0) \\ &= m \text{ or } m_1;\end{aligned}$$

2nd. $l = 2$, then

$$\begin{aligned}\phi(2m, 4) &= \phi(2m-2, 2) + \phi(2m-4, 2) + \dots + \phi(2, 2) \\ &= (m-1)_1 + (m-2)_1 + \dots + 1_1 \\ &= m_2;\end{aligned}$$

3rd. $l = 3$, then

$$\begin{aligned}\phi(2m, 6) &= \phi(2m-2, 4) + \phi(2m-4, 4) + \dots + \phi(4, 4) \\ &= (m-1)_2 + (m-2)_2 + \dots + 2_2 \\ &= m_3.\end{aligned}$$

In this way we see from equation (7) itself, that if it is true for any value of l , it is true for a value one greater and, therefore, true for all values. Hence

$$\phi(2m, 2l) = m_l, \quad (l = 1, 2 \dots m)$$

SYRACUSE UNIVERSITY, *February 10, 1898.*